

EXTENDED CESÁRO OPERATORS ON ZYGMUND SPACES IN THE UNIT BALL

ZHONG-SHAN FANG AND ZE-HUA ZHOU*

ABSTRACT. Let g be a holomorphic function of the unit ball B in the n -dimensional space, and denote by T_g and I_g the induced extended Cesáro operator and another integral operator. The boundedness and compactness of T_g and I_g acting on the Zygmund spaces in the unit ball are discussed and necessary and sufficient conditions are given in this paper.

1. INTRODUCTION

Let $f(z)$ be a holomorphic function on the unit disc D with *Taylor* expansion $f(z) = \sum_{j=0}^{\infty} a_j z^j$, the classical Cesáro operator acting on f is

$$\mathcal{C}[f](z) = \sum_{j=0}^{\infty} \left(\frac{1}{j+1} \sum_{k=0}^j a_k \right) z^j.$$

In the past few years, many authors focused on the boundedness and compactness of extended Cesáro operator between several spaces of holomorphic functions. It is well known that the operator \mathcal{C} is bounded on the usual Hardy spaces $H^p(D)$ for $0 < p < \infty$ and Bergman space, we recommend the interested readers refer to [10, 12, 8, 2, 13]. But the operator \mathcal{C} is not always bounded, in [15], Shi and Ren gave a sufficient and necessary condition for the operator \mathcal{C} to be bounded on mixed norm spaces in the unit disc. Recently, Siskakis and Zhao in [14] obtained sufficient and necessary conditions for Volterra type operator, which is a generalization of \mathcal{C} , to be bounded or compact between $BMOA$ spaces in the unit disc. It is a natural question to ask what are the conditions for higher dimensional case.

Let dv be the *Lebesgue* measure on the unit ball B of C^n normalized so that $v(B) = 1$, and $dv_{\beta} = c_{\beta}(1 - |z|^2)^{\beta} dv$, where c_{β} is a normalizing constant so that dv_{β} is a probability measure. The class of all holomorphic functions on B is defined by $H(B)$. For $f \in H(B)$ we write

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

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A little calculation shows $\mathcal{C}[f](z) = \frac{1}{z} \int_0^z f(t) (\log \frac{1}{1-t})' dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesàro operator (also called Volterra-type operator or Riemann-Stieltjes type operator) T_g on $H(B)$ defined by

$$T_g(f)(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t}.$$

It is easy to show that T_g take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesàro operator is bounded or compact.

Motivated by [15], Hu and Zhang [6, 7, 17] gave some sufficient and necessary conditions for the extended \mathcal{C} to be bounded and compact on mixed norm spaces, Bloch space as well as Dirichlet space in the unit ball.

Another natural integral operator is defined as follows:

$$I_g(f)(z) = \int_0^1 Rf(tz) g(tz) \frac{dt}{t}.$$

The importance of them comes from the fact that

$$(1) \quad T_g(f) + I_g(f) = M_g f - f(0)g(0)$$

where the multiplication operator is defined by

$$M_g(f)(z) = g(z)f(z), f \in H(B), z \in B.$$

Now we introduce some spaces first. Let H^∞ denote the space of all bounded holomorphic functions on the unit ball, equipped with the norm $\|f\|_\infty = \sup_{z \in B} |f(z)|$.

The Bloch space \mathcal{B} is defined as the space of holomorphic functions such that

$$\|f\|_{\mathcal{B}} = \sup\{(1 - |z|^2)|Rf(z)| : z \in B\} < \infty.$$

It is easy to check that if $f \in \mathcal{B}$ then

$$(2) \quad |f(z)| \leq C \log \frac{2}{1 - |z|^2} \|f\|_{\mathcal{B}}.$$

We define weighted Bloch space \mathcal{B}_{log} as the space of holomorphic functions $f \in H(B)$ such that

$$\|f\|_{\mathcal{B}_{log}} = \sup\{(1 - |z|^2)|Rf(z)| \log \frac{2}{1 - |z|^2} : z \in B\} < \infty.$$

The Zygmund space \mathcal{Z} [18] in the unit ball consists of those functions whose first order partial derivatives are in the Bloch space.

It is well known that (Theorem 7.11 in [18]) $f \in \mathcal{Z}$ if and only if $Rf \in \mathcal{B}$, and \mathcal{Z} is a Banach space with the norm

$$(3) \quad \|f\| = |f(0)| + \|Rf\|_{\mathcal{B}}.$$

The purpose of this paper is to discuss the boundedness and compactness of extended Cesàro operator T_g and another integral operator I_g on the Zygmund space in the unit ball.

2. SOME LEMMAS

In the following, we will use the symbol C to denote a finite positive number which does not depend on variable z and f .

In order to prove the main results, we will give some Lemmas first.

Lemma 1. *Assume $f \in \mathcal{Z}$, then we have*

$$\|f\|_\infty \leq C\|f\|$$

Proof. Since $f \in \mathcal{Z}$ implies that $Rf \in \mathcal{B}$, it follows from (2) that

$$(4) \quad |Rf(z)| \leq C \log \frac{2}{1-|z|^2} \|Rf\|_{\mathcal{B}} \leq C \log \frac{2}{1-|z|^2} \|f\|.$$

Furthermore by $\lim_{|z| \rightarrow 1} (1-|z|^2) \log \frac{2}{1-|z|^2} = 0$ we have

$$(5) \quad (1-|z|^2)|Rf(z)| \leq C(1-|z|^2) \log \frac{2}{1-|z|^2} \|f\| < \infty,$$

so $f \in \mathcal{B}$. It follows from Theorem 2.2 in [18] that

$$Rf(z) = \int_B \frac{Rf(w) dv_\beta(w)}{(1-\langle z, w \rangle)^{n+1+\beta}}$$

where β is a sufficiently large positive constant. Since $Rf(0) = 0$,

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt = \int_B Rf(w) L(z, w) dv_\beta(w)$$

where the kernel

$$L(z, w) = \int_0^1 \left(\frac{1}{(1-t\langle z, w \rangle)^{n+1+\beta}} - 1 \right) \frac{dt}{t}$$

satisfies

$$|L(z, w)| \leq \frac{C}{|1-\langle z, w \rangle|^{n+\beta}}$$

for all z and w in B . Note that $t^{1/2} \log \frac{2}{t} \leq \frac{2}{e} \cdot (1 - \log 2)$ for all $t \in (0, 1]$, then

$$\begin{aligned} |f(z) - f(0)| &= C \int_B \frac{(1-|w|^2)|Rf(w)| dv_{\beta-1}(w)}{|1-\langle z, w \rangle|^{n+\beta}} \\ &\leq C \int_B \frac{(1-|w|^2) \log \frac{2}{1-|w|^2} \|f\| dv_{\beta-1}(w)}{|1-\langle z, w \rangle|^{n+\beta}} \\ &\leq C \int_B \frac{(1-|w|^2)^{1-1/2} \|f\| dv_{\beta-1}(w)}{|1-\langle z, w \rangle|^{n+\beta}} \\ &\leq C \|f\|. \end{aligned}$$

The last inequality holds since $\int_B \frac{(1-|w|^2)^t dv(w)}{|1-\langle z, w \rangle|^{n+1+t+c}}$ is bounded for $c < 0$. This completes the proof of Lemma 1.

By Lemma 1, Montel theorem and the definition of compact operator, the following lemma follows.

Lemma 2. Assume that $g \in H(B)$. Then T_g (or I_g) : $\mathcal{Z} \rightarrow \mathcal{Z}$ is compact if and only if T_g (or I_g) is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in \mathcal{Z} which converges to zero uniformly on \overline{B} as $k \rightarrow \infty$, $\|T_g f_k\| \rightarrow 0$ (or $\|I_g f_k\| \rightarrow 0$) as $k \rightarrow \infty$.

Lemma 3. If $(f_k)_{k \in \mathbb{N}}$ is a bounded sequence in \mathcal{Z} which converges to zero uniformly on compact subsets of B as $k \rightarrow \infty$, then $\limsup_{k \rightarrow \infty} \sup_{z \in B} |f_k(z)| = 0$.

proof. Assume $\|f_k\| \leq M$. For any given $\epsilon > 0$, there exists $0 < \eta < 1$ such that $\frac{\sqrt{1-\eta}}{\eta} < \epsilon$. Note that $t^{1/2} \log \frac{2}{t} \leq \frac{2}{e} \cdot (1 - \log 2)$ for all $t \in (0, 1]$, then when $\eta < |z| < 1$, it follows from (4) that

$$\begin{aligned} |f_k(z) - f_k(\frac{\eta}{|z|}z)| &= \left| \int_{\frac{\eta}{|z|}}^1 Rf_k(tz) \frac{dt}{t} \right| \leq C \int_{\frac{\eta}{|z|}}^1 \log \frac{2}{1-|tz|^2} \|f_k\| \frac{dt}{t} \\ &\leq C \frac{|z|}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{\|f_k\| dt}{(1-|tz|^2)^{1/2}} \leq C \frac{M}{\eta} \int_{\frac{\eta}{|z|}}^1 \frac{|z| dt}{(1-t|z|)^{1/2}} \\ &\leq 2CM \frac{(1-\eta)^{1/2}}{\eta} < C\epsilon. \end{aligned}$$

So we get $\sup_{\eta < |z| < 1} |f_k(z)| \leq C\epsilon + \sup_{|w|=\eta} |f_k(w)|$. Thus, we have

$$\limsup_{k \rightarrow \infty} \sup_{z \in B} |f_k(z)| \leq \lim_{k \rightarrow \infty} (\sup_{|z| \leq \eta} |f_k(z)| + \sup_{\eta < |z| < 1} |f_k(z)|) \leq C\epsilon.$$

Now we finish the proof of this lemma.

Lemma 4. Let $g \in H(B)$, then

$$R[T_g f](z) = f(z)Rg(z)$$

for any $f \in H(B)$ and $z \in B$.

Proof. Suppose the holomorphic function fRg has the Taylor expansion

$$(fRg)(z) = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha.$$

Then we have

$$\begin{aligned} R(T_g f)(z) &= R \int_0^1 f(tz) R(tz) \frac{dt}{t} = R \int_0^1 \sum_{|\alpha| \geq 1} a_\alpha (tz)^\alpha \frac{dt}{t} \\ &= R \left[\sum_{|\alpha| \geq 1} \frac{a_\alpha z^\alpha}{|\alpha|} \right] = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha = (fRg)(z). \end{aligned}$$

3. MAIN THEOREMS

Theorem 1. Suppose $g \in H(B)$, then the following conditions are all equivalent:

- (a) T_g is bounded on \mathcal{Z} ;
- (b) T_g is compact on \mathcal{Z} ;

(c) $g \in \mathcal{Z}$.

Proof. $b \implies a$ is obvious. For $a \implies c$ we just take the test function given by $f(z) \equiv 1$.

We are going to prove $c \implies b$. Now assume that $g \in \mathcal{Z}$ and that $(f_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{Z} such that $\sup_{k \in \mathbb{N}} \|f_k\| \leq M$ and that $f_k \rightarrow 0$ uniformly on \overline{B} as $k \rightarrow \infty$. Now note that $T_g g_k(0) = 0$ and for every $\epsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$(1 - |z|^2) \left(\ln \frac{2}{1 - |z|^2} \right)^2 < \epsilon$$

whenever $\delta < |z| < 1$. Let $K = \{z \in B : |z| \leq \delta\}$, it follows from Lemma 4 and (4) that

$$\begin{aligned} \|T_g f_k\| &= \sup_{z \in B} (1 - |z|^2) |R(R(T_g f_k))| \\ &= \sup_{z \in B} (1 - |z|^2) |Rf_k \cdot Rg + f_k \cdot R(Rg)| \\ &\leq \sup_{z \in B} (1 - |z|^2) (|Rf_k \cdot Rg| + |f_k \cdot R(Rg)|) \\ &\leq \sup_{z \in K} (1 - |z|^2) |Rf_k \cdot Rg| + \sup_{z \in B-K} (1 - |z|^2) (|Rf_k \cdot Rg| \\ &\quad + \sup_{z \in B} (1 - |z|^2) |f_k \cdot R(Rg)|) \\ &\leq C \|g\| \sup_{z \in K} (1 - |z|^2) |Rf_k(z)| \log \frac{2}{1 - |z|^2} \\ &\quad + C \|f_k\| \cdot \|g\| \sup_{z \in B-K} (1 - |z|^2) \left(\log \frac{2}{1 - |z|^2} \right)^2 + \|g\| \cdot \sup_{z \in B} |f_k(z)|. \end{aligned}$$

With the uniform convergence of f_k to 0 and the Cauchy estimate, the conclusion follows by letting $k \rightarrow \infty$.

Theorem 2. Suppose $g \in H(B)$, $I_g : \mathcal{Z} \rightarrow \mathcal{Z}$. Then I_g is bounded if and only if $g \in H^\infty \cap \mathcal{B}_{\log}$.

Proof. First we assume that $g \in H^\infty \cap \mathcal{B}_{\log}$. Notice that $I_g f(0) = 0$ and $R(I_g f) = f Rg$, it follows from (4) that

$$\begin{aligned} (1 - |z|^2) |RR(I_g f)(z)| &= (1 - |z|^2) |R(Rf(z) \cdot g(z))| \\ &= (1 - |z|^2) |R(Rf)(z) \cdot g(z) + Rf(z) \cdot Rg(z)| \\ &\leq \|Rf(z)\|_{\mathcal{B}} \|g\|_\infty + |Rf(z)| (1 - |z|^2) |Rg(z)| \\ &\leq C \|f\| \cdot \|g\|_\infty + C \|f\| (1 - |z|^2) |Rg(z)| \log \frac{2}{1 - |z|^2} \\ &\leq C \|f\| \cdot \|g\|_\infty + C \|f\| \cdot \|g\|_{\mathcal{B}_{\log}}. \end{aligned}$$

The boundedness of I_g follows.

Conversely, assume that I_g is bounded, then there is a positive constant C such that

$$(6) \quad \|I_g f\| \leq C \|f\|$$

for every $f \in \mathcal{Z}$. Setting

$$h_a(z) = (\log \frac{2}{1-|a|^2})^{-1}(< z, a > -1)[(1 + \log \frac{2}{1- < z, a >})^2 + 1]$$

for $a \in B$ such that $|a| \geq \sqrt{1-2/e}$, then

$$Rh_a(z) = < z, a > (\log \frac{2}{1- < z, a >})^2 (\log \frac{2}{1-|a|^2})^{-1}$$

and

$$RRh_a(z) = \{ < z, a > (\log \frac{2}{1- < z, a >})^2 + \frac{2 < z, a >^2}{1- < z, a >} \log \frac{2}{1- < z, a >} \} (\log \frac{2}{1-|a|^2})^{-1}$$

It is easy to check that $M = \sup_{\sqrt{1-2/e} \leq |a| < 1} \|h_a\| < \infty$. Therefore, we have that

$$\begin{aligned} \infty &> \|I_g\| \|h_a\| \geq \|I_g h_a\| \\ &\geq \sup_{z \in B} (1 - |z|^2) |RRh_a(z) \cdot g(z) + Rh_a(z) \cdot Rg(z)| \\ &\geq (1 - |a|^2) \left| \frac{2|a|^4}{1-|a|^2} g(a) + |a|^2 \log \frac{2}{1-|a|^2} g(a) + |a|^2 Rg(a) \log \frac{2}{1-|a|^2} \right| \\ &\geq -\{2|a|^4 + |a|^2 \frac{2}{e} (1 - \log 2)\} |g(a)| + |a|^2 (1 - |a|^2) |Rg(a)| \log \frac{2}{1-|a|^2} \\ (7) &\geq -(2 + \frac{2}{e} (1 - \log 2)) |a|^2 + |a|^2 (1 - |a|^2) |Rg(a)| \log \frac{2}{1-|a|^2}. \end{aligned}$$

Next let

$$f_a(z) = h_a(z) - \int_0^1 < z, a > \log \frac{2}{1-t < z, a >} dt$$

then

$$Rf_a(z) = < z, a > \{ (\log \frac{2}{1- < z, a >})^2 (\log \frac{2}{1-|a|^2})^{-1} - \log \frac{2}{1- < z, a >} \}$$

$$RRf_a(z) = RRh_a(z) - < z, a > \log \frac{2}{1- < z, a >} - \frac{< z, a >^2}{1- < z, a >}$$

and consequently $N = \sup_{\sqrt{1-2/e} \leq |a| < 1} \|f_a\| < \infty$. Note that $Rf_a(a) = 0$ and

$RRf_a(a) = \frac{|a|^4}{1-|a|^2}$, we have

$$\begin{aligned} \infty &> \|I_g\| \cdot \|f_a\| \geq \|I_g f_a\| \\ &\geq \sup_{z \in B} (1 - |z|^2) |RRf_a(z) \cdot g(z) + Rf_a(z) \cdot Rg(z)| \\ (8) &\geq (1 - |a|^2) |RRf_a(a) g(a) + Rf_a(a) Rg(a)| = |a|^4 |g(a)|. \end{aligned}$$

From the maximum modulus theorem, we get $g \in H^\infty$. So it follows from (7) and (8) that

$$(9) \quad \sup_{\sqrt{1-2/e} \leq |a| < 1} (1 - |a|^2) |Rg(a)| \log \frac{2}{1-|a|^2} < \infty.$$

On the other hand, we have

$$\begin{aligned}
 & \sup_{|a| \leq \sqrt{1-2/e}} (1 - |a|^2) |Rg(a)| \log \frac{2}{1 - |a|^2} \\
 & \leq \frac{2}{e} \cdot (1 - \log 2) \max_{|a| = \sqrt{1-2/e}} |Rg(a)| \\
 (10) \quad & \leq \sup_{\sqrt{1-2/e} \leq |a| < 1} (1 - |a|^2) |Rg(a)| \log \frac{2}{1 - |a|^2} < +\infty.
 \end{aligned}$$

Combining (9) and (10), we finish the proof of Theorem 2.

Corollary The multiplication operator $M_g : \mathcal{Z} \rightarrow \mathcal{Z}$ is bounded if and only if $g \in \mathcal{Z}$.

Proof. If M_g is bounded on \mathcal{Z} , then setting the test function $f \equiv 1$, we have $M_g f = g \in \mathcal{Z}$.

Conversely, if $g \in \mathcal{Z}$, from Lemma 1 and (5), it is easy to see that $g \in H^\infty \cap \mathcal{B}_{\log}$, so by Theorems 1 and 2, both T_g and I_g are bounded, it follows from (1) that M_g is also bounded.

Theorem 3. Suppose $g \in H(B)$, $I_g : \mathcal{Z} \rightarrow \mathcal{Z}$. Then I_g is compact if and only if $g = 0$.

Proof. The sufficiency is obvious. We just need to prove the necessity. Suppose that I_g is compact, for any given sequence $(z_k)_{k \in \mathbb{N}}$ in B such that $|z_k| \rightarrow 1$ as $k \rightarrow \infty$, if we can show $g(z_k) \rightarrow 0$ as $k \rightarrow \infty$, then by the maximum modulus theorem we have $g \equiv 0$. In fact, setting

$$f_k(z) = h_{z_k}(z) - \left(\log \frac{2}{1 - |z_k|^2} \right)^{-2} \int_0^1 \langle z, z_k \rangle \left(\log \frac{2}{1 - t \langle z, z_k \rangle} \right)^3 dt.$$

Using the same way as in Theorem 2, we can show $\sup_{k \in \mathbb{N}} \|f_k\| \leq C$ and f_k converges to 0 uniformly on compact subsets of B . Since I_g is compact, we have $\|I_g f_k\| \rightarrow 0$ as $k \rightarrow \infty$. Note that $Rf_k(z_k) = 0$ and $RRf_k(z_k) = -\frac{|z_k|^4}{1 - |z_k|^2}$, it follows that

$$\begin{aligned}
 |z_k|^4 |g(z_k)| & \leq \sup_{z \in B} (1 - |z|^2) |RRf_k(z) \cdot g(z) + Rf_k(z) \cdot Rg(z)| \\
 & \leq \sup_{z \in B} (1 - |z|^2) |RR(I_g f_k)(z)| \leq \|I_g f_k\| \rightarrow 0
 \end{aligned}$$

as $k \rightarrow \infty$. This ends the proof of Theorem 3.

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DEPARTMENT OF MATHEMATICS
 TIANJIN POLYTECHNIC UNIVERSITY
 TIANJIN 300160
 P.R. CHINA.

E-mail address: fangzhongshan@yahoo.com.cn

DEPARTMENT OF MATHEMATICS
 TIANJIN UNIVERSITY
 TIANJIN 300072
 P.R. CHINA.

E-mail address: zehuazhou2003@yahoo.com.cn